Energy Budget of Fiber Bundle Model
Fiber bundle model (FBM)

- First used in textile engineering (Peirce, 1926)

- Statistical analysis (Daniels, 1945)

- Different load-sharing rules:
  - ELS, LLS, mixed-mode, hierarchical
Different ways of Loading

A) Quasi-static loading: weakest link failure

B) Loading by a fixed amount: expt. at lab

C) Overloaded situation: applied load > critical load
Problems in real-life
Static: Force-displacement

\[ F(x) = N[1 - P(x)]\kappa.x \]

\[ P(y) = \int_{0}^{y} p(x)dx \]

\[ \sigma = \frac{F(x)}{N} = [1 - P(x)]\kappa.x \]
Static: Critical strength

- F(x) has a maximum
- Put \( \frac{dF}{dx} = 0 \)

\[
1 - P(x_c) - x_c p(x_c) = 0
\]

(A) Uniform dist.

\[
P(x) = x; \quad x_c = \frac{1}{2}
\]

(B) Weibull dist.

\[
P(x) = 1 - e^{-x^k}; \quad x_c = k^{-\frac{1}{k}}
\]
Static: Burst or avalanche

\[ D(\Delta) \propto \Delta^{-5/2} \]

Hemmer & Hansen, 1992

\[ N = 10^6; \text{avg} = 20000 \]
Static: Crossover behavior

\[ D(\Delta) \propto \Delta^{-5/2} \left(1 - e^{-\Delta/\Delta_c}\right) \]

\[ \Delta_c = \frac{1}{8(x_c - x_0)^2} \]

\( N = 10^6; \text{avg} = 50000 \)

(Pradhan, Hemmer & Hansen; 2005)
Dynamics of fracture

Stress Field

changes

Fracture

changes

Crack-growth in nano-materials

......onset of earthquake
Dynamic: Solution of the breaking dynamics

- The recursive dynamics

\[ U_t = N_t / N \]

\[ U_{t+1} = 1 - P(\sigma / U_t) \]

Uniform dist.

\[ U_{t+1} = 1 - \sigma / U_t \]

Fixed point

\[ U_{t+1} = U_t = U^* \quad \rightarrow \quad U^{*2} - U^* + \sigma = 0 \]

Solution

\[ U^*(\sigma) = \frac{1}{2} \pm (\sigma_c - \sigma)^{1/2} \]

*(Pradhan, Bhattacharyya & Chakrabarti; 2001-02)*
Dynamic: Critical behavior

Order parameter

\[ O = U^*(\sigma) - U^*(\sigma_c) \approx (\sigma_c - \sigma)^{1/2} \]

Susceptibility

\[ \chi = \frac{dU^*(\sigma)}{d\sigma} \approx (\sigma_c - \sigma)^{-1/2} \]

Differential form

\[ \frac{dU}{dt} = U_t - U_{t+1} = U_t - 1 + \frac{\sigma}{U_t} \]

Relaxation time

\[ \tau \propto (\sigma_c - \sigma)^{-1/2} \]

(Pradhan, Bhattacharyya & Chakrabarti; 2001-02)
Above critical stress

\[ t_f(\sigma) \] is the failure step/time

Below critical stress

\[ t_f(\sigma) \] is the step/time to reach the fixed point
Exact solution: Above critical stress

\[ U_{t+1} = 1 - \frac{\sigma}{U_t} = 1 - \frac{1}{4} + \varepsilon \]

\[ U_t = \frac{1}{2} - y_t \sqrt{\varepsilon} \quad \Rightarrow \quad \frac{y_{t+1} - y_t}{1 + y_t y_{t+1}} = 2\sqrt{\varepsilon} \]

\[ y_t = \tan v_t \quad \Rightarrow \quad 2\sqrt{\varepsilon} = \frac{\tan v_{t+1} - \tan v_t}{1 + \tan v_{t+1} \tan v_t} = \tan(v_{t+1} - v_t) \]

\[ v_t = v_0 + t \tan^{-1}(2\sqrt{\varepsilon}) \]

solution

\[ t_f(\sigma) = \frac{\pi}{2} \left(\sigma - \sigma_c\right)^{-1/2} \]

Pradhan & Hemmer 2006
Exact Solution: Below critical stress

\[ U_{t+1} = 1 - \frac{\sigma}{U_t} = 1 - \frac{1}{4} \varepsilon \]

\[ U_t = \frac{1}{2} + \frac{\sqrt{\varepsilon}}{z_t} \quad \Rightarrow \quad \frac{z_{t+1} - z_t}{1 - z_t z_{t+1}} = 2\sqrt{\varepsilon} \]

\[ z_t = \tanh w_t \quad \Rightarrow \quad 2\sqrt{\varepsilon} = \frac{\tanh w_{t+1} - \tanh w_t}{1 - \tanh w_{t+1} \tanh w_t} = \tanh(w_{t+1} - w_t) \]

\[ w_t = w_0 + t \tanh^{-1}(2\sqrt{\varepsilon}) \]

solution

\[ t_f(\sigma) = \frac{\ln(N)}{4} (\sigma_c - \sigma)^{-1/2} \]

Critical amp. ratio \[ \frac{\ln(N)}{2\pi} \]

Pradhan & Hemmer 2006
Theory vs. Simulation

Below critical stress

Above critical stress

Pradhan & Hemmer 2006
\[ E^e(\Delta) = \frac{Nk}{2} \Delta^2 (1-P(\Delta)) \]

\[ E^d(\Delta) = \frac{Nk}{2} \int_0^\Delta x^2 p(x)dx \]
Energy Budget: Signal of upcoming failure

\[ \frac{dE^e}{d\Delta} \] has a maximum in the stable phase
Energy Budget: Signal of upcoming failure

\[ \Delta_c = G(\alpha) \Delta_{max} \]

\[ \Delta_c = H(k) \Delta_{max} \]
Energy Budget: Signal of upcoming failure

Power-law type dist.

\[ P(x) = x^{\alpha+1} \]

Weibull dist.

\[ P(x) = 1 - e^{-x^k} \]
Test-case: Mixed-distribution
Conclusions

- Elastic energy growth rate shows a peak before collapse point.

- Signal of upcoming failure can be used to predict collapse-situations (rock failure, mine-collapse, borehole collapse ...)

- The energy framework will help to formulate a thermodynamic description of FBM

- Can we calculate/check minimum entropy production principle?

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FBM research at PoreLab

- Field theory for ELS dynamics (Hendrick, Pradhan, Hansen, PRE 2018)
- Fracture front propagation during fluid injection (Pradhan et al.)
- Stretching of biological systems (Eivind et. al.)